Unidad 14 – Integrales indefinidas

ACTIVIDADES FINALES

EJERCICIOS Y PROBLEMAS

1. Resuelve las siguientes integrales por el método de integración de integrales inmediatas:

a) $\int (2x^2 - 4x + 5)\, dx$

b) $\int \left[\frac{x^4 - 3x\sqrt{x} + 2}{x}\right] dx$

c) $\int \frac{3x}{x^2+5} dx$

d) $\int \frac{2x}{\sqrt{3x^2+1}} dx$

e) $\int \cos\left(\frac{x}{2}\right) dx$

f) $\int \frac{dx}{4+7x^2}$

g) $\int \frac{3}{\sqrt{4-x^2}} dx$

h) $\int \left(3x + \frac{1}{x^2}\right) dx$

i) $\int \frac{(1+x)^2}{x} dx$

j) $\int \frac{4x+8}{x^2+4x} dx$

k) $\int \frac{\left(1+\sqrt{x}\right)^2}{\sqrt{x}} dx$

l) $\int 3x \cdot 3^{x^2}\, dx$

m) $\int \frac{x^3}{\sqrt{1-x^8}} dx$

n) $\int \operatorname{sen}^3 2x \cos 2x\, dx$

ñ) $\int \left(2\sqrt[4]{x^3} - \frac{5}{x}\right) dx$

o) $\int (2x^2+3)^2 \cdot 5x\, dx$

p) $\int 4x^2\sqrt{1-x^3}\, dx$

q) $\int \frac{1-\cos 2x}{2x - \operatorname{sen} 2x} dx$

r) $\int \frac{e^{\ln x}}{x} dx$

s) $\int \frac{x}{\sqrt{4-x^2}} dx$

t) $\int \frac{1-\ln x}{x \ln x} dx$

2. Resuelve las siguientes integrales por el método de integración por partes:

a) $\int x^2 \cdot \cos x\, dx$

b) $\int e^x \cdot \cos 2x\, dx$

c) $\int \operatorname{arcsen} x\, dx$

d) $\int x \operatorname{sen} x \cdot \cos x\, dx$

e) $\int x^3 \cdot \ln x\, dx$

f) $\int 2^x \cdot \operatorname{sen} x\, dx$

g) $\int \operatorname{arctg} x\, dx$

h) $\int x^3 \ln^2 x\, dx$

i) $\int x^2 \cdot e^x\, dx$

j) $\int \ln x\, dx$

k) $\int \sqrt{x} \cdot \ln x\, dx$

l) $\int \frac{x \operatorname{arcsen} x}{\sqrt{1-x^2}} dx$

3. Resuelve las siguientes integrales por el método de integración de funciones racionales:

a) $\int \frac{x}{x-2} dx$

b) $\int \frac{x^2+x}{(1-x)(1+x^2)} dx$

c) $\int \frac{dx}{x^2+2x+1}$

d) $\int \frac{9x}{x^3+5x^2+8x+4} dx$

e) $\int \frac{dx}{x^3-3x^2+2x}$

f) $\int \frac{-x^2+6x-1}{(x-1)^2(x+1)} dx$

g) $\int \frac{x^3+4x}{x^2+1} dx$

h) $\int \frac{x}{(x-1)^2} dx$

i) $\int \frac{x^3}{x^2-1} dx$

j) $\int \frac{3x^2+5x-7}{x^3-2x^2+x-2} dx$

k) $\int \frac{x^4+2x-6}{x^2+x-2} dx$

l) $\int \frac{x^3}{x^2+1} dx$

SOLUCIONES

1. Las integrales quedan:

a) $\int (2x^2 - 4x + 5)\, dx = \frac{2x^3}{3} - 2x^2 + 5x + C$

b) $\int \frac{x^4\, 3x\sqrt{x+2}}{x}\, dx = \int \left(x^3 - 3x^{1/2} + \frac{2}{x} \right) dx = x^4 / 4 - 2\sqrt{x^3}\; 2\ln|x| + C$

c) $\int \frac{3x}{x^2+5}\, dx = \frac{3}{2} \int \frac{2x}{x^2+5}\, dx = \frac{3}{2} \ln|x^2+5| + C$

d) $\int \frac{2x}{\sqrt{3x^2+1}}\, dx = \frac{1}{3} \int (3x^2+1)^{-\frac{1}{2}} \cdot 6x \cdot dx =$

$= \frac{2}{3} \sqrt{3x^2+1} + C$

e) $\int \cos\left(\frac{x}{2}\right) dx = 2 \int \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} \cdot dx = 2\ \text{sen}\left(\frac{x}{2}\right) + C$

f) $\int \frac{dx}{4+7x^2} = \frac{1}{4} \int \frac{1}{1 + \left(\frac{\sqrt{7}\, x}{2}\right)^2}\, dx =$

$= \frac{1}{4} \cdot \frac{2}{\sqrt{7}} \int \frac{\frac{\sqrt{7}}{2}}{1 + \left(\frac{\sqrt{7}\, x}{2}\right)^2}\, dx =$

$= \frac{1}{2\sqrt{7}} \cdot arc\ tg\left(\frac{\sqrt{7}\, x}{2}\right) + C$

g) $\int \frac{3}{\sqrt{4-x^2}}\, dx = 3 \int \frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{x}{2}\right)^2}}\, dx =$

$= 3 \cdot arc\ sen\left(\frac{x}{2}\right) + C$

h) $\int\left(3x+\frac{1}{x^2}\right)dx=\int\left(3x+x^{-2}\right)dx=\frac{3x^2}{2}-1/x+C$

i) $\int \frac{(1+x)^2}{x}\,dx = \int\left(x+2+\frac{1}{x}\right)dx =$

$= \frac{x^2}{2}+2x+ln\,|x|+C$

j) $\int\frac{4x+8}{x^2+4x}dx=2\int\frac{2x+4}{x^2+4x}dx=2\ln\left|x^2+4x\right|+C$

k) $\int\frac{\left(1+\sqrt{x}\right)^2}{\sqrt{x}}dx=2\int\frac{1}{2\sqrt{x}}\left(1+\sqrt{x}\right)^2dx=\frac{2}{3}\cdot\left(1+\sqrt{x}\right)^3+C$

l) $\int 3x\cdot 3^{x^x}\,dx=\frac{3}{2}\int 3^{x^2}\cdot 2x=\frac{3}{2}\cdot\frac{3^{x^2}}{\ln 3}+C$

m) $\int\frac{x^3}{\sqrt{1-x^8}}dx=\frac{1}{4}\int\frac{4x^3}{1-(x^4)^2}dx=\frac{1}{4}\text{arc tg}(x^4)+C$

n) $\int \text{sen}^3 2x\cdot\cos 2x\cdot d=\frac{1}{2}\int(sen\,2x)^3\cdot 2\cdot cos\,2x\,dx=\frac{1}{2}\cdot\frac{(sen\,2x)^4}{4}+C$

ñ) $\int\left(2\sqrt[4]{x^3}-\frac{5}{x}\right)dx=\int\left(2x^{\frac{3}{4}}-\frac{5}{x}\right)dx=\frac{8\sqrt[4]{x^7}}{7}-5\,ln\,|x|+C$

o) $\int(2x^2+3)^2\cdot 5x\,dx=\frac{5}{4}\int(2x^2+3)^2\cdot 4x\,dx=\frac{5}{4}\,\frac{(2x^2+3)^3}{3}+C$

p) $\int 4x^2\cdot\sqrt{1-x^3}\,dx=\frac{4}{-3}\int(1-x^3)^{\frac{1}{2}}\cdot(-3x^2)\,dx=$

$-\frac{4}{3}\,\frac{2\sqrt{(1-x^3)^3}}{3}=\frac{-8}{9}\sqrt{(1-x^3)^3}+C$

q) $\displaystyle\int \frac{1 - \cos 2x}{2x - \operatorname{sen} 2x}\, dx = \frac{1}{2} \int \frac{2 - 2\cos 2x}{2x - \operatorname{sen} 2x}\, dx = \frac{1}{2} \ln |2x - \operatorname{sen} 2x| + C$

r) $\displaystyle\int \frac{e^{\ln x}}{x} \cdot dx = e^{\ln x} + C$

s) $\displaystyle\int \frac{x}{\sqrt{4 - x^2}}\, dx = \frac{1}{-2} \int (4 - x^2)^{\frac{1}{2}} \cdot (-2x)\, dx =$

$$-\frac{1}{2}\,\frac{(4 - x^2)^{1/2}}{\frac{1}{2}} = -\sqrt{4 - x^2} + C$$

t) $\displaystyle\int \frac{1 - \ln x}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx - \int \frac{1}{x}\, dx =$

$$= \ln |\ln x| - \ln x + C$$

2. Las integrales quedan:

a) $\int x^2 \cdot \cos x \, dx = I$

$$\left.\begin{array}{r} u = x^2 \Rightarrow du = 2x \cdot dx \\ dv = \cos x \cdot dx \Rightarrow v = \operatorname{sen} x \end{array}\right\}$$

$$I = \int x^2 \cdot \cos x \cdot dx = x^2 \cdot \operatorname{sen} x - \int 2x \cdot \operatorname{sen} x \cdot dx$$

Aplicamos de nuevo el método de integración por partes:

$$\left.\begin{array}{r} u = 2x \Rightarrow du = 2 \cdot dx \\ dv = \operatorname{sen} x \cdot dx \Rightarrow v = -\cos x \end{array}\right\}$$

$$I = x^2 \operatorname{sen} x - \left[-2x \cos x - \int 2 \cdot (-\cos x) \cdot dx\right] =$$

$$= x^2 \cdot \operatorname{sen} x + 2x \cos x - 2 \operatorname{sen} x + C$$

Por tanto:

$$\int x^2 \cdot \cos x \, dx = x^2 \cdot \operatorname{sen} x + 2x \cdot \cos x - 2 \operatorname{sen} x + C$$

b) $\int e^x \cdot \cos 2x \cdot dx = I$

$$\left.\begin{array}{r} u = \cos 2x \Rightarrow du = -2 \operatorname{sen} 2x \, dx \\ dv = e^x \cdot dx \Rightarrow v = e^x \end{array}\right\}$$

$$I = \int e^x \cdot \cos 2x \cdot dx = e^x \cdot \cos 2x - \int -2 e^x \operatorname{sen} 2x \, dx =$$

$$= e^x \cos 2x + 2 \int e^x \cdot \operatorname{sen} 2x \cdot dx$$

Aplicamos de nuevo el método de integración por partes:

$$\left.\begin{array}{r} u = \operatorname{sen} 2x \Rightarrow du = 2 \cdot \cos 2x \cdot dx \\ dv = e^x \cdot dx \Rightarrow v = e^x \end{array}\right\}$$

$$I = \int e^x \cdot \cos 2x + 2 \left[e^x \operatorname{sen} 2x - \int 2 e^x \cos 2x \, dx\right] =$$

$$= e^x \cos 2x + 2 e^x \operatorname{sen} 2x - 4 \int e^x \cdot \cos 2x \, dx \Rightarrow$$

$$\Rightarrow I = e^x \cdot \cos 2x + 2 e^x \cdot \operatorname{sen} 2x - 4 I \Rightarrow$$

$$\Rightarrow I = \frac{e^x \cos 2x + 2 e^x \operatorname{sen} 2x}{5} + C$$

$$\left.\begin{array}{r} u = \operatorname{arc\,sen} x \Rightarrow du = \dfrac{1}{\sqrt{1 - x^2}} \, dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$$I = \int \operatorname{arc\,sen} x \cdot dx = x \cdot \operatorname{arc\,sen} x - \int \frac{x}{\sqrt{1 - x^2}} \cdot dx =$$

c) $\int$ arc sen $x \cdot dx = I$

$$\left.\begin{array}{r} u = arc\ sen\ x \Rightarrow du = \dfrac{1}{\sqrt{1-x^2}}\ dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$$I = \int arc\ sen\ x \cdot dx = x \cdot arc\ sen\ x - \int \frac{x}{\sqrt{1-x^2}} \cdot dx =$$

$$= x \cdot arc\ sen\ x + \frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} (-2x)\ dx =$$

$$= x \cdot arc\ sen\ x + \sqrt{1-x^2} + C$$

d) $\int x \cdot sen\ x \cdot cos\ x \cdot dx = I = \int \frac{x}{2} \cdot sen\ 2x \cdot dx$

$$\left.\begin{array}{r} u = \dfrac{x}{2} \Rightarrow du = \dfrac{1}{2}\ dx \\ dv = sen\ 2x \cdot dx \Rightarrow v = \dfrac{-cos\ 2x}{2} \end{array}\right\}$$

e) $\int x^3 \cdot \ln x \cdot dx = I$

$$\left.\begin{array}{r} u = ln\ x \Rightarrow du = \dfrac{1}{x}\ dx \\ dv = x^3\ dx \Rightarrow v = \dfrac{x^4}{4} \end{array}\right\}$$

$$I = \int x^3 \cdot ln\ x \cdot dx = \frac{x^4}{4}\ ln\ x - \int \frac{x^4}{4x}\ dx =$$

$$= \frac{x^4}{4}\ ln\ x - \frac{x^4}{16} + C$$

f) $\int 2^x \cdot$ sen $x \cdot dx = I$

$$\left.\begin{array}{r} u = 2^x \Rightarrow du = 2^x \cdot ln\ 2 \cdot dx \\ dv = sen\ x \cdot dx \Rightarrow v = -cos\ x \end{array}\right\}$$

$$I = \int 2^x \cdot sen\ x \cdot dx = -2^x \cdot cos\ x + \int 2^x \cdot ln\ 2 \cdot cos\ x \cdot dx$$

Aplicando de nuevo este método, obtenemos:

$$\left.\begin{array}{l} u = 2^x \Rightarrow du = 2^x \cdot \ln 2 \cdot dx \\ dv = \cos x \cdot dx \Rightarrow v = \operatorname{sen} x \end{array}\right\}$$

$$I = \int -2^x \cdot \cos x +$$

$$+ \ln 2 \left[2^x \cdot \operatorname{sen} x - \int 2^x \cdot \ln 2 \cdot \operatorname{sen} x \cdot dx \right] = -2^x \cdot \cos x +$$

$$+ 2^x \cdot \ln 2 \cdot \operatorname{sen} x - (\ln 2)^2 \int 2^x \cdot \operatorname{sen} x \cdot dx \Rightarrow$$

$$\Rightarrow I = -2^x \cdot \cos x + 2^x \cdot \ln 2 \cdot \operatorname{sen} x - (\ln 2)^2 \cdot I \Rightarrow$$

$$\Rightarrow I = \frac{-2^x \cdot \cos x + 2^x \cdot \ln 2 \cdot \operatorname{sen} x}{1 + (\ln 2)^2} + C$$

g) $\int \text{arc tg}\, x \cdot dx = I$

$$\left.\begin{array}{r} u = \text{arc tg}\, x \Rightarrow du = \dfrac{1}{1 - x^2}\, dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$$I = \int \text{arc tg}\, x\, dx = x \cdot \text{arc tg}\, x - \int \frac{x}{1 + x^2}\, dx =$$

$$= x \cdot \text{arc tg}\, x - \frac{1}{2} \int \frac{2x}{1 + x^2}\, dx =$$

$$= x \cdot \text{arc tg}\, x - \frac{1}{2} \ln |1 + x^2| + C$$

h) $\int x^3 \cdot \ln^2 x \cdot dx = I$

$$\left.\begin{array}{r} u = \ln^2 x \Rightarrow du = 2 \ln x \cdot \dfrac{1}{x} \cdot dx \\ dv = x^3\, dx \Rightarrow v = \dfrac{x^4}{4} \end{array}\right\}$$

$$I = \int x^3 \cdot \ln^2 x \cdot dx = \frac{x^4}{4} \ln^2 x - \int \frac{x^4}{4} \cdot 2 \ln x \cdot \frac{1}{x}\, dx =$$

$$= \frac{x^4}{4} \ln^2 x - \frac{1}{2} \int x^3 \cdot \ln x \cdot dx$$

Aplicando este método a la última integral, obtenemos:

$$\left.\begin{array}{r} u = \ln x \Rightarrow du = \dfrac{1}{x}\, dx \\ dv = x^3\, dx \Rightarrow v = \dfrac{x^4}{4} \end{array}\right\}$$

$$I = \frac{x^4}{4} \ln^2 x - \frac{1}{2}\left[\frac{x^4}{4} \ln x - \int \frac{x^4}{4} \cdot \frac{1}{x} \cdot dx\right] =$$

$$= \frac{x^4}{4} \ln^2 x - \frac{x^4}{8} \ln x + \frac{x^4}{32} + C$$

i) $\int x^2 \cdot e^x \cdot dx = I$

$$\left.\begin{array}{r} u = x^2 \Rightarrow du = 2x \cdot dx \\ dv = \cos x \cdot dx \Rightarrow v = \operatorname{sen} x \end{array}\right\}$$

$$I = \int x^2 \cdot e^x \cdot dx = x^2 \cdot e^x - \int 2x \cdot e^x \cdot dx \Rightarrow$$

Aplicamos de nuevo el método de integración por partes:

$$\left.\begin{array}{l} u = 2x \Rightarrow du = 2\, dx \\ dv = e^x\, dx \Rightarrow v = e^x \end{array}\right\}$$

$$I = \int x^2 \cdot e^x \cdot dx = x^2\, e^x - \left[2x\, e^x - \int 2\, e^x\, dx\right] =$$

$$= x^2 \cdot e^x - 2xe^x + 2e^x + C \Rightarrow$$

$$\int x^2 \cdot e^x \cdot dx = x^2 \cdot e^x - 2x\, e^x + 2\, e^x + C$$

j) $\int \ln x \cdot dx = I$

$$\left.\begin{array}{r} u = \ln x \Rightarrow du = \dfrac{1}{x}\, dx \\ dv = dx \Rightarrow v = x \end{array}\right|$$

$$I = \int \ln x \cdot dx = x \cdot \ln x - \int x \cdot \frac{1}{x} \cdot dx = x \cdot \ln x - x \Rightarrow$$

$$\Rightarrow \int \ln x \cdot dx = x \ln x - x + C$$

k) $\int \sqrt{x} \cdot \ln x \cdot dx = I$

$$\left.\begin{array}{l} u = \ln x \Rightarrow du = \dfrac{1}{x}\, dx \\ dv = \sqrt{x} \cdot dx \Rightarrow v = \dfrac{2\sqrt{x^3}}{3} \end{array}\right\}$$

$$I = \int \sqrt{x} \cdot \ln x \, dx = \frac{2\sqrt{x^3}}{3} \ln x - \int \frac{2}{3} x^{\frac{1}{2}} \, dx =$$

$$= \frac{2\sqrt{x^3}}{3} \ln x - \frac{4}{9} \sqrt{x^3} + C$$

l) $\int \frac{x \cdot \operatorname{arc\,sen} x}{\sqrt{1-x^2}} dx = I$

$$\left.\begin{array}{l} u = \operatorname{arc\,sen} x \Rightarrow du = \dfrac{1}{\sqrt{1-x^2}}\, dx \\ dv = \dfrac{x}{\sqrt{1-x^2}}\, dx \Rightarrow v = -\sqrt{1-x^2} \end{array}\right\}$$

$$I = \int \frac{x \cdot \operatorname{arc\,sen} x}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2} \cdot \operatorname{arc\,sen} x -$$

$$- \int \frac{-\sqrt{1-x^2}}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2} \cdot \operatorname{arc\,sen} x + x + C$$

3. Las integrales quedan:

a) $\int \frac{x}{x-2}\,dx = \int \frac{x-2}{x-2}\,dx + \int \frac{2}{x-2}\,dx =$

$= x + 2\ ln\ |x-2| + C$

b) $\int \frac{x^2+x}{(1-x)(1+x^2)}dx$

Descomponemos la fracción en suma de fracciones simples:

$$\frac{x^2+x}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}$$

$$\frac{x^2+x}{(1-x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$$

- $x = 1 \Rightarrow 2A = 2 \Rightarrow A = 1$
- $x = 0 \Rightarrow A + C = 0 \Rightarrow C = -1$
- $x = -1 \Rightarrow 2A - 2B + 2C = 0 \Rightarrow B = 0$

La integral pedida vale:

$$\int \frac{x^2+x}{(1-x)(1+x^2)}\,dx = \int \frac{1}{1-x}\,dx + \int \frac{-1}{1+x^2}\,dx =$$

$= -ln\ |1-x| - arc\ tg\ x + C$

c) $\int \frac{1}{x^2+2x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$

d) $\int \frac{9x}{x^3+5x^2+8x+4}\,dx = -9\int \frac{1}{x+1}\,dx +$

$+ 9\int \frac{1}{x+2}\,dx + 18\int \frac{1}{(x+2)^2}\,dx =$

$= -9\ ln\ |x+1| + 9\ ln\ |x+2| - \frac{18}{x+2} + C$

e) $\int \frac{dx}{x^3 - 3x^2 + 2x}$

Descomponemos la fracción integrando en suma de fracciones simples:

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\frac{1}{x(x-1)(x-2)} =$$

$$= \frac{A(x-1)(x-2) + B \cdot x \cdot (x-2) + C \cdot x(x-1)}{x(x-1)(x-2)}$$

- $x = 1 \Rightarrow -B = 1 \Rightarrow B = -1$
- $x = 0 \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$
- $x = 2 \Rightarrow 2C = 1 \Rightarrow C = \frac{1}{2}$

La integral pedida vale:

$$\int \frac{dx}{x^3 - 3x^2 + 2x} = \frac{1}{2} \int \frac{1}{x}\, dx - \int \frac{1}{x-1}\, dx +$$

$$+ \frac{1}{2} \int \frac{1}{x-2}\, dx = \frac{1}{2}\, ln\, |x| - ln\, |x-1| +$$

$$+ \frac{1}{2}\, ln\, |x-2| + C = ln \left| \frac{\sqrt{x(x-2)}}{x-1} \right| + C$$

f) $\int \frac{-x^2+6x-1}{(x-1)^2 \cdot (x+1)} dx$

Descomponemos la fracción en suma de fracciones simples:

$$\frac{-x^2+6x-1}{(x-1)^2 \cdot (x+1)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x+1}$$

$$\frac{-x^2+6x-1}{(x-1)^2 \cdot (x+1)} =$$

$$= \frac{A(x+1) + B(x-1)(x+1) + C(x-1)^2}{(x-1)^2 \cdot (x+1)}$$

- $x = 1 \Rightarrow 2A = 4 \Rightarrow A = 2$
- $x = -1 \Rightarrow 4C = -8 \Rightarrow C = -2$
- $x = 0 \Rightarrow A - B + C = -1 \Rightarrow B = 1$

La integral pedida vale:

$$\int \frac{-x^2+6x-1}{(x-1)^2 \cdot (x+1)} dx = \int \frac{2}{(x-1)^2} dx + \int \frac{1}{x-1} dx +$$

$$+ \int \frac{-2}{x+1} dx = -\frac{2}{x-1} + ln\,|x-1| - 2\,ln\,|x+1| + C =$$

$$= \frac{-2}{x-1} + ln \left| \frac{x-1}{(x+1)^2} \right| + C$$

g) $\int \frac{x^3+4x}{x^2+1} dx = \int x dx + \frac{3}{2} \int \frac{2x}{x^2+1} dx = \frac{x^2}{2} +$

$$\frac{3}{2}\, ln\,(x^2+1) + C$$

h) $\int \frac{x}{(x-1)^2} dx = \frac{1}{2} \int \frac{2x}{(x-1)^2} dx =$

$$= \frac{1}{2} \int \frac{2x-2+2}{(x-1)^2} dx = \frac{1}{2} \int \frac{2(x-1)}{(x-1)^2} dx +$$

$$+ \int \frac{1}{(x-1)^2} dx = ln\,|x-1| - \frac{1}{x-1} + C$$

i) $\int \frac{x^3}{x^2-1}dx = \int \frac{x(x^2-1)+x}{x^2-1} =$

$= \int x\,dx + \int \frac{x}{x^2-1}\,dx = \frac{x^2}{2} + \frac{1}{2}\,ln\,|x^2-1| + C$

j) $\int \frac{3x^2+5x-7}{x^3-2\cdot x^2+x-2}dx$

Descomponemos la fracción dada en suma de fracciones simples:

$$\frac{3x^2+5x-7}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

$$\frac{3x^2+5x-7}{(x-2)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x-2)}{(x-2)(x^2+1)}$$

- $x = 2 \Rightarrow 5A = 15 \Rightarrow A = 3$
- $x = 0 \Rightarrow A - 2C = -7 \Rightarrow C = 5$
- $x = 1 \Rightarrow 2A - B - C = 1 \Rightarrow B = 0$

La integral pedida vale:

$$\int \frac{3x^2+5x-7}{(x-2)(x^2+1)}\,dx = \int \frac{3}{x-2}\,dx + \int \frac{5}{x^2+1}\,dx =$$

$$= 3\,ln\,|x-2| + 5\cdot arc\,tg\,x + C$$

k) $\int \frac{x^4+2x-6}{x^2+x-2}dx = \int \left[x^2-x+3-\frac{3x}{x^2+x-2}\right]dx =$

$= \int x^2\,dx - \int x dx + 3\int dx - \int \frac{1}{x-1}\,dx -$

$-2\int \frac{1}{x+2}\,dx = \frac{x^3}{3} - \frac{x^2}{2} +$

$+ 3x - ln\,|x-1| - 2\,ln\,|x+2| + C$

l) $\int \frac{x^3}{1+x^2}dx = \int \left[x - \frac{x}{x^2+1}\right]dx =$

$= \int xdx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{x^2}{2} - \frac{1}{2}\,ln\,(x^2+1) + C$

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4. Resuelve las siguientes integrales por el método de integración de cambio de variable con el cambio que se indica en cada caso:

a) $\int \frac{e^x}{e^{2x}+e^x+2}dx \quad [e^x = t]$

b) $\int \frac{1}{x+\sqrt{x}}dx \quad [\sqrt{x} = t]$

c) $\int \frac{x}{\sqrt{4-x^2}}dx \quad [4 - x^2 = t^2]$

d) $\int \frac{x^3}{\sqrt{x-1}}dx \quad [\sqrt{x-1} = t]$

e) $\int \cos^{-4} x \, dx \quad [\text{tg } x = t]$

f) $\int \frac{3^x + 27^x}{1+9^x}dx \quad [3^x = t]$

5. Resuelve las siguientes integrales por el método de integración de cambio de variable:

a) $\int x\sqrt{x-1}dx$

b) $\int \frac{\sqrt{2x-3}}{\sqrt{2x-3}+1}dx$

c) $\int \frac{dx}{x \cdot \ln^2 x}$

d) $\int \frac{e^{-x}}{1+e^{-x}}dx$

e) $\int \frac{dx}{(x+5)\sqrt{x+1}}$

f) $\int \frac{\text{sen}3x}{\sqrt[3]{1+3\cos 3x}}dx$

g) $\int \frac{\sqrt[3]{1+\ln x}}{x}dx$

h) $\int \frac{\sqrt{x}}{x+2}dx$

i) $\int \frac{\sqrt{x^2+1}}{x}dx$

6. Resuelve las siguientes integrales por el método de integración más conveniente:

a) $\int \frac{3}{1+\sqrt{x+1}}dx$

b) $\int \text{sen}^3 x \, dx$

c) $\int \frac{[\ln x]^5}{x}dx$

d) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

e) $\int \text{sen}(\ln x) \cdot dx$

f) $\int \frac{6x^3 - x}{1+x^4}dx$

g) $\int x \cdot \ln\left[\frac{1-x}{1+x}\right]dx$

h) $\int \text{sen}^4 5x \cdot \cos 5x \, dx$

i) $\int x^2 \cdot \text{arcsen } x \, dx$

j) $\int \frac{\text{arcsen } x}{\sqrt{1-x^2}}dx$

k) $\int \frac{dx}{x[\ln x - 1]}$

l) $\int \frac{\ln(\ln x)}{x}dx$

m) $\int \frac{\text{sen } x}{\cos^2 x}dx$

n) $\int x \ln(x^2 - 1) \, dx$

ñ) $\int \frac{4x^2}{x^4 - 1}dx$

o) $\int \frac{\cos 5x}{\text{sen}^4 5x}dx$

p) $\int \frac{1}{\sqrt{1+4x-x^2}}dx$

q) $\int \frac{3x}{x^4+16}dx$

r) $\int \frac{\sqrt{x}+\ln x}{2x}dx$

s) $\int \frac{x^4 - 8}{x^3 - 4x}dx$

t) $\int \ln\left[x+\sqrt{1+x^2}\right]dx$

u) $\int \frac{6x^3 - 7x}{\sqrt{1-x^4}}dx$

v) $\int \frac{2+x^2-3x}{(1+x^2)}dx$

w) $\int \sqrt{6-5x^2} \, dx$

7. Halla la primitiva de la función $f(x) = \frac{\sqrt{x^2-1}}{x}$ cuya gráfica pase por el punto (2, 2).

4. Las integrales son:

a) $\int \frac{e^x}{e^{2x}+e^x+2}\,dx = \int \frac{1}{t^2+t+2}\,dt =$

$$= \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{7}{4}} = \frac{7}{4}\int \frac{1}{1+\left[\frac{2}{\sqrt{7}}\left(t+\frac{1}{2}\right)\right]^2}\,dt =$$

$$= \frac{7\sqrt{7}}{8}\int \frac{1}{1+\left(\frac{2}{\sqrt{7}}\,t+\frac{1}{\sqrt{7}}\right)^2}\,\frac{2}{\sqrt{7}}\,dt =$$

$$= \frac{7\sqrt{7}}{8}\,arc\,tg\left(\frac{2}{\sqrt{7}}\,e^x+\frac{1}{\sqrt{7}}\right)+C$$

b) $\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{1}{t^2+t}\,2t\,dt = 2\int \frac{1}{t+1}\,dt =$

$= 2\ ln\,|t+1] + C = 2\ ln\,|\sqrt{x}+1| + C$

c) $\int \frac{x}{\sqrt{4-x^2}}\,dx = -\int dt\ = -t + C = -\sqrt{4-x^2} + C$

d) $\int \frac{x^3}{\sqrt{x-1}}\,dx = 2\int (t^2+1)^3\,dt =$

$$= 2\int [t^6+3t^4+3t^2+1]\,dt = \frac{2t^7}{7}+\frac{6t^5}{5}+2t^3+2t+C =$$

$$= \frac{2}{7}\,(\sqrt{x}\ -1)^7 + \frac{6}{5}\,(\sqrt{x}\ -1)^5 + 2\,(\sqrt{x}\ -1)^3 +$$

$$+ 2\,\sqrt{x\ -1} + C$$

e) $\int \frac{1}{\cos^4 x}\,dx = \int \frac{1}{\cos^2 x}\cdot\frac{1}{\cos^2 x}\,dx =$

$= \int (1 + tg^2 x)\cdot\frac{1}{\cos^2 x}\,dx = \int (1 + t^2)\,dt =$

$= t + \frac{t^3}{3} + C = tgx + \frac{tg^3 x}{3} + C$

f) $\int \frac{3^x + 27^x}{1 + 9^x}\,dx = \int \frac{3^x(1 + 9^x)}{1 + 9^x}\,dx = \int 3^x\,dx =$

$= \frac{1}{\ln 3}\int dt = \frac{1}{\ln 3}\,t + C = \frac{1}{\ln 3}\,3^x + C$

5. Las integrales son:

a) $\int x\sqrt{x-1}\,dx$ Hacemos el cambio de variables:

$x - 1 = t^2 \Rightarrow dx = 2t\,dt$

$\int x\cdot\sqrt{x-1}\,dx = \int (t^2 + 1)\cdot t\cdot 2t\,dt =$

$\int (2t^4 + 2t^2)\,dt = \frac{2t^5}{5} + \frac{2t^3}{3} + C$

Deshaciendo el cambio $t=\sqrt{x-1}$, obtenemos:

$\int x\sqrt{x-1}\,dx = 2\,\frac{\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + C$

b) $\int \frac{\sqrt{2x-3}}{\sqrt{2x-3+1}}\,dx$

Hacemos el cambio de variable: $2x-3=t^2 \Rightarrow dx = t\,dt$.

$\int \frac{\sqrt{2x-3}}{\sqrt{2x-3}+1}\,dx = \int \frac{t}{t+1}\cdot t\,dt = \int \frac{t^2}{t+1}\,dt =$

$= \int \frac{(t+1)(t-1)+1}{t}\,dt = \int (t-1)\,dt +$

$+ \int \frac{1}{t+1}\,dt = \frac{t^2}{2} - t + \ln|t+1| + C$

Deshaciendo el cambio: $t = \sqrt{2x-3}$, obtenemos:

$$\int \frac{\sqrt{2x-3}}{\sqrt{2x-3}+1}\,dx = \frac{2x-3}{2} - \sqrt{2x-3} +$$

$$+ \ln|\sqrt{2x-3}+1| + C$$

c) $\int \frac{dx}{x \cdot \ln^2 x} = \int (\ln x)^{-2} \cdot \frac{1}{x} \cdot dx = \frac{(\ln x)^{-1}}{-1} = -1/\ln x + C$

d) $\int \frac{e^{-x}}{1+e^{-x}}\,dx = -\ln\left|1+e^{-x}\right| + C$

También se puede hacer mediante el cambio de variable: $1+e^{-x} = t$.

e) $\int \frac{dx}{(x+5)\sqrt{x+1}}$ hacemos el cambio: $x+1=t^2 \Rightarrow dx = 2t\,dt$

$$\int \frac{dx}{(x+5)\sqrt{x+1}} = \int \frac{2t\,dt}{(t^2+4)\cdot t} = \int \frac{2}{t^2+4}\,dt =$$

$$\int \frac{1/2}{1+t^2/4}\,dt = \int \frac{1/2}{1+(t/2)^2}\,dt = arc\ tg\left(\frac{t}{2}\right) + C =$$

$= arc\ tg\,\frac{\sqrt{x+1}}{2} + C$ tras deshacer el cambio con

$t = \sqrt{x+1}$.

f) $$\int \frac{sen\,3x}{\sqrt[3]{1+3\cos 3x}}\,dx = \int (1+3\cos 3x)^{-\frac{1}{3}} \cdot$$

$$\cdot\, sen\,3x \cdot dx = \frac{1}{-9}\int (1+3\cos 3x)^{-\frac{1}{3}} \cdot$$

$$\cdot\,(-9\,sen\,3x)\,dx = -\frac{1}{9}\,\frac{(1+3\cos 3x)^{2/3}}{2/3} =$$

$$= \frac{-\sqrt[3]{(1+3\cos 3x)^2}}{6} + C$$

También se puede hacer mediante el cambio de variable: $1+3\cos 2x = t^3$.

g) $\int \frac{\sqrt[3]{1+\ln x}}{x}dx = -\int (1+\ln x)^{1/3} \cdot \frac{1}{x} \cdot dx =$

$= \frac{(1 + \ln x)^{4/3}}{4/3} = \frac{3}{4}\sqrt[3]{(1 + \ln x)^4} + C$

También se puede hacer con el cambio de variable: $1 + \ln x = t$.

h) $\int \frac{\sqrt{x}}{x+2}dx$

Hacemos el cambio: $x = t^2 \Rightarrow dx = 2t\,dt$

$\int \frac{\sqrt{x}}{x + 2}\,dx = \int \frac{t}{t^2 + 2} \cdot 2t \cdot dt = \int \frac{2t^2}{t^2 + 2}\,dt =$

$= 2\int \frac{(t^2 + 2) - 2}{t^2 + 2}\,dt = 2\int \frac{t^2 + 2}{t^2 + 2}\,dt -$

$- 4\int \frac{1}{t^2 + 2}\,dt = 2t - 2\int \frac{1}{1 + \frac{t^2}{2}}\,dt =$

$= 2t - 2\int \frac{1}{1 + \left(\frac{t}{\sqrt{2}}\right)^2}\,dt = 2t - 2 \cdot \sqrt{2}\int \frac{\frac{1}{\sqrt{2}}}{1 + \left(\frac{t}{\sqrt{2}}\right)^2}\,dt =$

$= 2t - 2\sqrt{2} \cdot \operatorname{arc\,tg}\left(\frac{t}{\sqrt{2}}\right) = 2\sqrt{x} - 2\sqrt{2} \cdot \operatorname{arc\,tg}\sqrt{\frac{x}{2}} + C$

al deshacer el cambio con $t = \sqrt{x}$.

i) $\int \frac{\sqrt{x^2+1}}{x}dx =$

Hacemos el cambio $x^2 + 1 = t^2 \Rightarrow dx = \frac{t\,dt}{x}$

$\int \frac{\sqrt{x^2 + 1}}{x}\,dx = \int \frac{t}{x} \cdot \frac{t\,dt}{x} = \int \frac{t^2}{x^2}\,dt = \int \frac{t^2}{t^2 - 1}\,dt =$

$= \int \frac{t^2 - 1 + 1}{t^2 - 1}\,dt = \int \frac{t^2 - 1}{t^2 - 1}\,dt + \int \frac{1}{t^2 - 1}\,dt =$

$= t + \int \frac{1/2}{t - 1}\,dt + \int \frac{-1/2}{t + 1}\,dt = t + \frac{1}{2}\ln|t - 1| -$

$- \frac{1}{2}\ln|t + 1| + C = t + \ln\left|\sqrt{\frac{t - 1}{t + 1}}\right| + C = \sqrt{x^2 + 1} +$

$+ \ln\sqrt{\frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 1} + 1}} + C$

6. Las integrales quedan:

a) $\int \frac{3}{1+\sqrt{x+1}}dx$

La resolveremos por el método de cambio de variable haciendo: $x+1=t^2 \Rightarrow dx=2t\,dt$

$$3\int \frac{1}{1+t}\cdot 2t\,dt = 3\int \frac{2(t+1)-2}{t+1}\,dt =$$

$$= 3\int 2\,dt - 3\int \frac{2}{t+1}\,dt = 6t - 6\,ln\,|t+1| =$$

$$= 6\sqrt{x+1} - 6\,ln\,|\sqrt{x+1}+1| + C$$

b) $\int sen^3 x\cdot dx = \int sen\,x\cdot sen\,x^2\cdot dx =$

$$= \int sen\,x\,(1-cos^2 x)\,dx = \int sen\,x\,dx -$$

$$-\int (cos\,x)^2\cdot sen\,x\cdot dx = -cos\,x + \int (cos\,x)^2\,(-sen\,x)\,dx =$$

$$= -cos\,x + \frac{(cos\,x)^3}{3} + C$$

c) $\int \frac{(\ln x)^5}{x}dx = \int (\ln x)^5\cdot\frac{1}{x}\cdot dx = \frac{(\ln x)^6}{6}+C$

d) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$ hacemos esta integral por el método de cambio de variable, haciendo:

$x^2+2x=t \Rightarrow dx = \frac{t\,dt}{x+1}$

$$\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{1}{(x+1)\cdot t}\cdot\frac{t\,dt}{x+1} =$$

$$= \int \frac{1}{(x+1)^2}\,dt = \int \frac{1}{x^2+2x+1}\,dt =$$

$$= \int \frac{1}{t^2+1}\,dt = arc\,tg\,t = arc\,tg\,\sqrt{x^2+2x} + C$$

e) $\int sen(\ln x)\,dx = I$

Esta integral la resolvemos por el método de integración por partes.

Esta integral la resolvemos por el método de integración por partes.

$$\left.\begin{array}{r} u = sen\ (ln\ x) \Rightarrow du = cos\ (ln\ x) \cdot \dfrac{1}{x} \cdot dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$$I = \int sen\ (ln\ x)\ dx = x \cdot sen\ (ln\ x) -$$

$$-\int x \cdot cos\ (ln\ x) \cdot \frac{1}{x} \cdot dx = x \cdot sen\ (ln\ x) - \int cos\ (ln\ x)\ dx$$

Volvemos a aplicar este método a la última integral:

$$\left.\begin{array}{r} u = cos\ (ln\ x) \Rightarrow du = -sen\ (ln\ x) \cdot \dfrac{1}{x} \cdot dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$$I = x \cdot sen\ (ln\ x) - \left[x \cdot cos\ (ln\ x) - \int -x\ sen\ (ln\ x) \cdot \frac{1}{x}\ dx\right] =$$

$$= x \cdot sen\ (ln\ x) - x\ cos\ (ln\ x) - \int sen\ (ln\ x)\ dx \Rightarrow$$

$$\Rightarrow I = x\ sen\ (ln\ x) - x\ cos\ (ln\ x) - I \Rightarrow 2I = x\ sen\ (ln\ x) -$$

$$- x\ cos\ (ln\ x) \Rightarrow I = \frac{x\ sen\ (ln\ x) - x\ cos\ (ln\ x)}{2} + C \Rightarrow$$

$$\Rightarrow \int sen\ (ln\ x)\ dx = \frac{x\ sen\ (ln\ x) - x\ cos\ (ln\ x)}{2} + C$$

f) $\displaystyle \int \frac{6x^3 - x}{1 + x^4}\ dx = \int \frac{6x^3}{1 + x^4}\ dx - \int \frac{x}{1 + x^4}\ dx =$

$$= \frac{6}{4} \int \frac{4x^3}{1 + x^4}\ dx - \frac{1}{2} \int \frac{2x}{1 + (x^2)^2}\ dx =$$

$$= \frac{3}{2}\ ln\ |1 + x^4| - \frac{1}{2}\ arc\ tg\ (x^2) + C$$

g) $\int x \cdot \ln\left(\frac{1-x}{1+x}\right) dx = I$

Hacemos esta integral por medio del método de integración por partes:

$$\left.\begin{array}{r} u = \ln\left(\frac{1-x}{1+x}\right) \Rightarrow du = \frac{-2}{1-x^2}\,dx \\ dv = x \cdot dx \Rightarrow v = \frac{x^2}{2} \end{array}\right\}$$

$$I = \int x \cdot \ln\left(\frac{1-x}{1+x}\right) dx = \frac{x^2}{2} \cdot \ln\left(\frac{1-x}{1+x}\right) -$$

$$-\int \frac{x^2}{2} \cdot \frac{-2}{1-x^2}\,dx = \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) -$$

$$-\int \frac{x^2}{x^2-1}\,dx = \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) - \int \frac{x^2-1+1}{x^2-1}\,dx =$$

$$= \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) - \int \frac{x^2-1}{x^2-1}\,dx - \int \frac{1}{x^2-1}\,dx =$$

$$= \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) - x - \underset{(*)}{\int \frac{1}{x^2-1}\,dx} =$$

$$= \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) - x - \int \frac{1/2}{x-1}\,dx - \int \frac{-1/2}{x+1}\,dx +$$

$$+ \frac{x^2}{2}\ln\left(\frac{1-x}{1+x}\right) - x - \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

(*) en esta integral hemos aplicado el método de integración de funciones racionales, descomponiendo la fracción $\frac{1}{x^2-1}$ en suma de fracciones simples:

$$\frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

h) $\int sen^4 5x \cdot \cos 5x \cdot dx = \frac{1}{5}\int (sen 5x)^{-4} \cdot 5 \cdot \cos 5x \cdot dx =$

$= \frac{1}{5} \frac{(sen\ 5x)^{-3}}{-3} = \frac{-1}{15\ (sen\ 5x)^3} + C$

i) $\int x^2 \cdot arc\ sen\ x \cdot dx = I$

La resolvemos por el método de integración por partes:

$$\left.\begin{array}{r} u = arc\ sen\ x \Rightarrow du = \dfrac{1}{\sqrt{1-x^2}}\ dx \\ dv = x^2\ dx \Rightarrow v = \dfrac{x^3}{3} \end{array}\right\}$$

$$I = \int x^2 \cdot arc\ sen\ x \cdot dx = \frac{x^3}{3}\ arc\ sen\ x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\ dx$$

Esta última integral la resolvemos por cambio de variables, haciendo $1 - x^2 = t^2 \Rightarrow dx = \frac{-t\ dt}{x}$

$$\int \frac{x^3}{\sqrt{1-x^2}}\ dx = \int \frac{x^3}{t} \cdot \frac{-t\ dt}{x} \int -x^2 \cdot dt =$$

$$= \int (t^2 - 1)\ dt = \frac{t^3}{3} - t = \frac{\sqrt{(1-x^2)^3}}{3} - \sqrt{1-x^2}$$

Por tanto, la integral pedida vale:

$$\int x^2 \cdot arc\ sen \cdot dx = \frac{x^3}{x} \cdot arc\ sen\ x -$$

$$- \frac{1}{3}\left[\frac{\sqrt{(1-x^2)^3}}{3} - \sqrt{1-x^2}\right] = \frac{x^3}{3}\ arc\ sen\ x -$$

$$- \frac{\sqrt{(1-x^2)^3}}{9} + \frac{\sqrt{1-x^2}}{3} + C$$

j) $\int \frac{arc\ sen\ x}{\sqrt{1-x^2}}\ dx = \int (arc\ sen\ x) \cdot \frac{1}{\sqrt{1-x^2}}\ dx = \frac{(arc\ sen\ x)^2}{2} + C$

k) $\int \dfrac{dx}{x(\ln x - 1)} = \int \dfrac{1/x \cdot dx}{\ln x - 1} = \ln|\ln x - 1| + C$

l) $\int \dfrac{\ln(\ln x)}{x} dx = I$

$$I = \int \frac{\ln t}{x} \cdot x\, dt = \int \ln t \cdot dt$$

Hacemos esta integral por el método de cambio de variable, haciendo: $\ln x = t \Rightarrow dx = x \cdot dt$

$$\left.\begin{array}{r} u = \ln t \Rightarrow du = \dfrac{1}{t}\, dt \\ dv = dt \Rightarrow v = t \end{array}\right\}$$

$$\int \ln t \cdot dt = t \cdot \ln t - \int t \cdot \frac{1}{t} \cdot dt = t \cdot \ln t - t \Rightarrow$$

$$\Rightarrow I = \int \frac{\ln(\ln x)}{x} \cdot dx = \ln x \cdot [\ln(\ln x)] - \ln x + C$$

m) $\int \dfrac{sen\, x}{\cos^2 x} dx = -\int \dfrac{1}{t^2} dt = \dfrac{1}{t} + C = \dfrac{1}{\cos x} + C$

Hemos realizado el cambio de variable $\cos x = t$

n) $\int x \cdot \ln(x^2 - 1) \cdot dx = I$

Esta integral la resolvemos por el método de integración por partes:

$$\left.\begin{array}{r} u = \ln(x^2 - 1) \Rightarrow du = \dfrac{2x}{x^2 - 1}\, dx \\ dv = x\, dx \Rightarrow v = \dfrac{x^2}{2} \end{array}\right\}$$

$$I = \int x \cdot \ln(x^2 - 1) \cdot dx = \frac{x^2}{2} \ln(x^2 - 1) - \int \frac{x^3}{x^2 - 1} dx =$$

$$= \frac{x^2}{2} \ln(x^2 - 1) - \int \frac{(x^2 - 1)\, x + x}{x^2 - 1} dt = \frac{x^2}{2} \ln(x^2 - 1) -$$

$$- \int x\, dx - \frac{1}{2} \int \frac{2x}{x^2 - 1} dx = \frac{x^2}{2} \ln(x^2 - 1) - \frac{x^2}{2} -$$

$$- \frac{1}{2} \ln|x^2 - 1| + C$$

ñ) $\int \frac{4x^2}{x^4-1}dx = \int \frac{1}{x-1}dx - \int \frac{1}{x+1}dx +$

$+ 2\int \frac{1}{1+x^2}\,dx = ln\,|x-1| - ln\,|x+1| + 2\,arc\,tg\,x + C.$

o) $\int \frac{\cos 5x}{sen^4 5x}\,dx = \frac{1}{5}\int (sen\,5x)^{-4}\cdot 5\cdot \cos 5x\cdot dx \;\; = \frac{1}{5}\,\frac{(sen\,5x)^{-3}}{-3} = \frac{-1}{15\,(sen\,5x)^3} + C$

p) $\int \frac{1}{\sqrt{1+4x-x^2}}\,dx = \int \frac{1}{\sqrt{5-(2-x)^2}}\,dx =$

$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-\frac{(2-x)^2}{5}}}\,dx =$

$= -\int \frac{\frac{-1}{\sqrt{5}}}{\sqrt{1-\left(\frac{2-x}{\sqrt{5}}\right)^2}}\,dx = -arc\;sen\left(\frac{2-x}{\sqrt{5}}\right) + C$

q)

$\int \frac{3x}{x^4+16}\,dx = \int \frac{\frac{3x}{16}}{1+\frac{x^4}{16}}\,dx = \int \frac{\frac{3x}{16}}{1+\left(\frac{x^2}{4}\right)^2}\,dx =$

$= \frac{3}{8}\int \frac{\frac{x}{2}}{1+\left(\frac{x^2}{4}\right)^2}\,dx = \frac{3}{8}\cdot arc\;tg\left(\frac{x^2}{4}\right) + C$

r) $\int \frac{\sqrt{x}+ln\,x}{2x}\,dx = \int \frac{\sqrt{x}}{2x}\,dx + \frac{1}{2}\int \frac{ln\,x}{x}\,dx =$

$= \frac{1}{2}\int x^{-1/2}\,dx + \frac{1}{2}\int ln\,x\cdot\frac{1}{x}\cdot dx = \frac{1}{2}\cdot\frac{x^{1/2}}{1/2} +$

$+ \frac{1}{2}\,\frac{(ln\,x)^2}{2} + C = \sqrt{x} + \frac{(ln\,x)^2}{4} + C$

s) $\int \frac{x^4-8}{x^3-4x}\,dx = \int \left[x + \frac{4x^2-8}{x^3-4x}\right] dx =$

$= \int x dx + 2\int \frac{1}{x}\,dx + \int \frac{1}{x-2}\,dx + \int \frac{1}{x+2}\,dx =$

$= \frac{x^2}{2} + 2\ ln\ |x| + ln\ |x-2| + ln\ |x+2| + C$

t) $\int \ln(x+\sqrt{1+x^2})\,dx = 1$

Hacemos esta integral por el método de integración por partes:

$$\left.\begin{array}{r} u = ln\ (x+\sqrt{1+x^2}) \Rightarrow du = \frac{1}{\sqrt{1+x^2}} \cdot dx \\ dv = dx \Rightarrow v = x \end{array}\right\}$$

$I = \int ln\ (x+\sqrt{1+x^2}) \cdot dx = x \cdot ln\ (x+\sqrt{1+x^2}) -$

$- \int \frac{x}{\sqrt{1+x^2}}\,dx = x \cdot ln\ (x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

u) $\int \frac{6x^3-7x}{\sqrt{1-x^4}}\,dx = \int \frac{6x^3}{\sqrt{1-x^4}}\,dx - \int \frac{7x}{\sqrt{1-x^4}}\,dx =$

$= \frac{6}{-4}\int (1-x^4)^{-\frac{1}{2}} \cdot (-4x^3)\,dx - \frac{7}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx =$

$= -\frac{3}{2}\ \frac{(1-x^4)^{1/2}}{1/2} - \frac{7}{2}\ arc\ sen\ (x^2) =$

$= -3\sqrt{1-x^4} - \frac{7}{2} \cdot arc\ sen\ (x^2) + C$

v) $\int \frac{x^2-3x+2}{x^2+1}\,dx = \int \left[1 + \frac{-3x+1}{x^2+1}\right] dx =$

$= \int dx - \frac{3}{2}\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx =$

$= x - \frac{3}{2}\ ln\ (x^2+1) + arc\ tg\ x + C$

w) $\int \sqrt{6-5x^2}\, dx$

Esta integral la resolvemos por el método de integración por cambio de variable, haciendo:

$$x = \frac{\sqrt{6} \cdot sen\, t}{\sqrt{5}} \Rightarrow dx = \frac{\sqrt{6} \cdot cos\, t}{\sqrt{5}} \cdot dt$$

$$\int \sqrt{6-5x^2} \cdot dx = \int \sqrt{6 - 5 \cdot \frac{6\, sen^2\, t}{5}} \cdot \frac{\sqrt{6} \cdot cos\, t}{\sqrt{5}} \cdot dt =$$

$$= \int \frac{6}{\sqrt{5}}\, cos^2\, t\, dt = \frac{6}{\sqrt{5}} \int \frac{1 + cos\, 2t}{2}\, dt = \frac{3}{\sqrt{5}} \cdot t +$$

$$+ \frac{3}{2\sqrt{5}}\, sen\, 2t = \frac{3}{\sqrt{5}} \cdot arc\, sen\, \frac{\sqrt{5}\, x}{\sqrt{6}} +$$

$$+ \frac{3}{2\sqrt{5}} \cdot 2 \cdot sen\, t \cdot cos\, t = \frac{3}{\sqrt{5}} \cdot arc\, sen\, \frac{\sqrt{5}\, x}{\sqrt{6}} +$$

$$+ \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}\, x}{\sqrt{6}} \sqrt{1 - \left(\frac{\sqrt{5}\, x}{\sqrt{6}}\right)^2} + C$$

7. Calculamos las primitivas de f(x):

Resolvemos esta integral por el método de la integración de cambio de variable, haciendo: $x^2 - 1 = t^2 \Rightarrow dx = \frac{t\, dt}{x}$.

$$\int \frac{\sqrt{x^2-1}}{x} \cdot dx = \int \frac{t}{x} \cdot \frac{t\, dt}{x} = \int \frac{t^2}{t^2+1}\, dt =$$

$$= \int \frac{t^2+1-1}{t^2+1}\, dt = \int \frac{t^2+1}{t^2+1}\, dt - \int \frac{1}{t^2+1}\, dt =$$

$$= t - arc\, tg\, t = \sqrt{x^2-1} - arc\, tg\, \sqrt{x^2-1} + C$$

Todas las primitivas de $f(x)$ son las funciones:

$$F(x) = \sqrt{x^2-1} - arc\, tg\, \sqrt{x^2-1} + C$$

La primitiva buscada que pase por el punto (2, 2) cumple:

$$2 = \sqrt{3} - arc\, tg\, \sqrt{3} + C \Rightarrow C = 2 - \sqrt{3} + \frac{\pi}{3}$$

Luego la primitiva buscada es:

$$F(x) = \sqrt{x^2-1} - arc\, tg\, \sqrt{x^2-1} + \left(2 - \sqrt{3} + \frac{\pi}{3}\right)$$

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8. Calcula:

a) $\int x^3 \cdot e^{x^2}\, dx$

b) $\int \frac{e^{2x}}{2+e^x} dx$

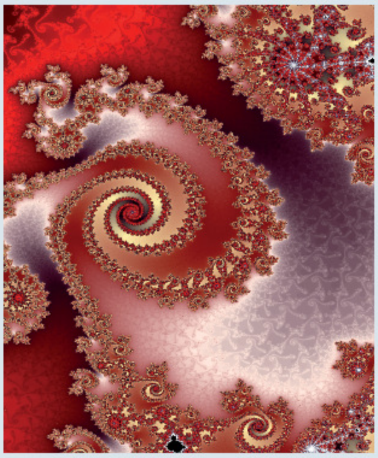
9. Resuelve: $\int \cos^3 x \cdot \text{sen}^2 x\, dx$.

10. Calcula las siguientes integrales indefinidas:

$I_1 = \int e^{3x} \cos 2x\, dx$ $\quad I_2 = \int x\, e^{-x}\, dx$ $\quad I_3 = \int x^2 \,\text{sen}\, x\, dx$

11. Resuelve la siguiente integral indefinida:

$$I = \int \frac{x^3 - x}{x^2 + 4x - 12} dx$$



12. Calcula:

$$I = \int \frac{x+1}{x^2 - x} dx$$

13. Resuelve las siguientes integrales indefinidas:

$I_1 = \int \frac{x-1}{x^2 + 2x + 3} dx$ $\quad I_2 = \int \frac{5x+8}{2x^2 + x - 3} dx$

14. Resuelve $\int \frac{4^x + 5 \cdot 16^x}{1 + 16^x} dx$.

15. Calcula $\int \frac{1 + \ln x}{x(\ln^2 x - \ln x)} dx$.

16. Calcula la primitiva de la función $f(x) = [\ln x]^2$ que se anule en $x = e$.

17. Calcula, integrando por partes, las siguientes integrales. Comprueba el resultado por derivación.

$I_1 = \int x \cdot \text{sen}\,(\ln x)\, dx$ $\quad I_2 = \int x \cdot \ln (x^2 + 1)\, dx$ $\quad I_3 = \int x^2 \cdot \ln (2x + 1)\, dx$

18. Halla $f(x)$ si sabemos que $f(0) = 1$; $f'(0) = 2$ y $f''(x) = 3x$.

19. Calcula una función real $f: \mathbb{R} \to \mathbb{R}$ que cumple las condiciones siguientes:

$f'(0) = 5, \quad f''(0) = 1, \quad f(0) = 0 \quad \text{y} \quad f'''(x) = x + 1$

8. Las integrales quedan:

a) Haciendo el cambio de variable $x^2 = t$ obtenemos:

$\int x^3 \cdot e^{x^2} dx = \frac{1}{2}\int t \cdot e^t dt$ resolviendo esta integral por partes obtenemos:

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2}\int t \cdot e^t dt = \frac{1}{2} e^{x^2}\left(x^2 - 1\right) + C$$

b) Haciendo el cambio de variable $e^x = t$ obtenemos:

$$\int \frac{e^{2x}}{2+e^x} dx = \int \frac{t}{2+t} dt = \int \frac{2+t}{2+t} dt - \int \frac{2}{2+t} dt = t - 2\ln(2+t) = e^x - 2\ln\left(2+e^x\right) + C$$

9. La solución queda:

$$\int \cos^3 x \cdot \text{sen}^2 x \cdot dx = \int \cos x \cdot \left(1 - \text{sen}^2 x\right) \cdot \text{sen}^2 x \cdot dx = \int \left(\text{sen}^2 x - \text{sen}^4 x\right) \cdot \cos x \cdot dx = \frac{\text{sen}^3 x}{3} - \frac{\text{sen}^5 x}{5} + C$$

10. Todas las integrales pueden calcularse utilizando integración por partes. Obtenemos:

$$I_1 = \int e^{3x} \cos 2x \, dx = \frac{1}{2} e^{3x} \text{sen } 2x - \frac{3}{2}\int e^{3x} \text{sen } 2x \, dx =$$

$$= \frac{1}{2} e^{3x} \text{sen } 2x - \frac{3}{2}\left\{- \frac{1}{2} e^{x} \cos 2x + \right.$$

$$\left. + \frac{3}{2} \int e^{3x} \cos 2x \, dx \right\}$$

$$I_1 = \frac{2}{13} e^{3x} \text{sen } 2x + \frac{3}{13} e^{3x} \cos 2x \, dx + C$$

$$I_2 = \int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C$$

$$I_3 = \int x^2 \text{sen } x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx =$$

$$= -x^2 \cos x + 2x \text{ sen } x - 2 \int \text{sen } x \, dx =$$

$$= -x^2 \cos x + 2x \text{ sen } x + 2 \cos x + C$$

11. La solución es:

$$I = \int \frac{x^3 - x}{x^2 + 4x - 12}\,dx = \int \left[x - 4 + \frac{27x - 48}{x^2 + 4x - 12} \right] dx =$$

$$= \int x\,dx - 4\int dx + \int \frac{27x - 48}{x^2 + 4x - 12}\,dx =$$

$$= \int x\,dx - 4\int dx + \frac{3}{4}\int \frac{1}{x-2}\,dx + \frac{105}{4}\int \frac{1}{x+6}\,dx =$$

$$= \frac{x^2}{2} - 4x + \frac{3}{4}\,ln\,|x-2| + \frac{105}{4}\,ln\,|x+6| + C.$$

12. La integral es:

$\int \frac{x+1}{x^2 - x} dx$ Descomponemos la fracción $\frac{x+1}{x^2 - x}$ en suma de fracciones simples:

$$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow$$

$$\Rightarrow \frac{x+1}{x(x-1)} = \frac{-1}{x} + \frac{2}{x-1}$$

$$\int \frac{x+1}{x^2 - x}\,dx = \int \frac{-1}{x}\,dx + \int \frac{2}{x-1}\,dx =$$

$$= -\,ln\,|x| + 2\,ln\,|x-1| + C$$

13. La solución en cada caso es:

$$I_1 = \int \frac{x-1}{x^2+2x+3}\,dx = \int \frac{x}{x^2+2x+3}\,dx -$$

$$-\int \frac{1}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+3}\,dx -$$

$$-\int \frac{1}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx -$$

$$-2\int \frac{1}{(x+1)^2+2}\,dx = \frac{1}{2}\,ln\,|x^2+2x+3| -$$

$$-\sqrt{2}\int \frac{1\cdot\frac{1}{\sqrt{2}}}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}\,dx =$$

$$= \frac{1}{2}\,ln\,|x^2+2x+3| - \sqrt{2}\cdot arc\,tg\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$$I_2 = \int \frac{5x+8}{2x^2+x-3}\,dx = \int \left[\frac{13/5}{x-1} - \frac{1/5}{2x+3}\right]dx =$$

$$= \frac{13}{5}\int \frac{1}{x-1}\,dx - \frac{1}{10}\int \frac{2}{2x+3}\,dx =$$

$$= \frac{13}{5}\,ln\,|x-1| - \frac{1}{10}\,ln\,|2x+3| + C$$

14. Quedan:

$$\int \frac{4^x+5\cdot 16^x}{1+16^x}\,dx = \int \frac{4^x}{1+16^x}\,dx + \int \frac{5\cdot 16^x}{1+16^x}\,dx =$$

$$= \int \frac{4^x}{1+(4^x)^2}\,dx + 5\int \frac{16^x}{1+16^x}\,dx =$$

$$= \frac{1}{ln\,4}\int \frac{4^x\cdot ln\,4}{1+(4^x)^2}\,dx + \frac{5}{ln\,16}\int \frac{16^x\cdot ln\,16}{1+16^x}\,dx =$$

$$= \frac{1}{ln\,4}\cdot arc\,tg\,(4^x) + \frac{5}{ln\,16}\cdot ln\,|1+16^x| + C$$

15. Haciendo $t = \ln x, dt = \frac{dx}{x}$ la integral queda:

$$\int \frac{1 + \ln x}{x (\ln^2 x - \ln x)} \, dx = \int \frac{1 + t}{t^2 - t} \, dt =$$

$$= \int \left[-\frac{1}{t} + \frac{2}{t - 1} \right] dt = -\int \frac{1}{t} \, dt + 2 \int \frac{1}{t - 1} \, dt =$$

$$= -\ln |t| + 2 \ln |t - 1| + C = -\ln |\ln x| +$$

$$+ 2 \ln |\ln x - 1| + C.$$

16. $\int (\ln x)^2 \, dx = I$

$$\left. \begin{array}{r} u = (\ln x)^2 \Rightarrow du = 2 (\ln x) \cdot \frac{1}{x} \cdot dx \\ dv = dx \Rightarrow v = x \end{array} \right\}$$

$$I = \int (\ln x)^2 \, dx = x \cdot (\ln x)^2 - \int 2 \ln x \cdot dx$$

$$\left. \begin{array}{r} u = 2 \ln x \Rightarrow du = \frac{2}{x} \, dx \\ dv = dx \Rightarrow v = x \end{array} \right\}$$

$$I = \int (\ln x)^2 \, dx = x \cdot (\ln x)^2 - \left[2x \ln x - \int 2 \, dx \right] =$$

$$= x (\ln x)^2 - 2x \ln x + 2x + C$$

Todas las primitivas de f(x) = $(\ln x)^2$ son las funciones de la forma:

$$F(x) = x (\ln x)^2 - 2x \ln x + 2x + C$$

Lo que se anula para $x = e$ verificara: $0 = e - 2e + 2e + C \Rightarrow C = -e$

La primitiva buscada es:

$$F(x) = x (\ln x)^2 - 2x \ln x + 2x - e$$

17. Las integrales quedan del siguiente modo:

$$I_1 = \int x \cdot \text{sen}(\ln x)\, dx$$

$$\left.\begin{array}{r} u = sen\,(ln\,x) \Rightarrow du = cos\,(ln\,x) \cdot \dfrac{1}{x} \cdot dx \\ dv = x\,dx \Rightarrow v = \dfrac{x^2}{2} \end{array}\right\}$$

$$I_1 = \int x \cdot sen\,(ln\,x)\,dx = \frac{x^2}{2}\,sen\,(ln\,x) -$$

$$-\int \frac{x}{2} \cdot cos\,(ln\,x)\,dx$$

Esta última integral la resolvemos por el mismo método de integración por partes:

$$\left.\begin{array}{r} u = cos\,(ln\,x) \Rightarrow du = -sen\,(ln\,x) \cdot \dfrac{1}{x} \cdot dx \\ dv = \dfrac{x}{2}\,dx \Rightarrow v = \dfrac{x^2}{4} \end{array}\right\}$$

$$I_1 = \int x \cdot sen\,(ln\,x)\,dx = \frac{x^2}{2}\,sen\,(ln\,x) -$$

$$-\left[\frac{x^2}{4}\,cos\,(ln\,x) - \int -\frac{x}{4}\,sen\,(ln\,x)\,dx\right] \Rightarrow$$

$$\Rightarrow I_1 = \frac{x^2}{2}\,sen\,(ln\,x) - \frac{x^2}{4}\,cos\,(ln\,x) - \frac{1}{4}\,I_1$$

$$\frac{5}{4}\,I_1 = \frac{x^2}{2}\,sen\,(ln\,x) - \frac{x^2}{4}\,cos\,(ln\,x) \Rightarrow$$

$$\Rightarrow I_1 = \frac{4}{5}\left[\frac{x^2}{2}\,sen\,(ln\,x) - \frac{x^2}{4}\,cos\,(ln\,x)\right] + C$$

$$I_1 = \int x \cdot sen\,(ln\,x)\,dx = \frac{2x^2\,sen\,(ln\,x)}{5} -$$

$$-\frac{x^2\,cos\,(ln\,x)}{5} + C$$

Vamos a comprobar el resultado, para ello veremos que la derivada del segundo miembro es igual a la función del primer miembro:

$$D\left[\frac{2x^2 \cdot sen\,(ln\,x)}{5} - \frac{x^2 \cdot cos\,(ln\,x)}{5} + C\right] =$$

$$= \frac{4x \cdot sen\,(ln\,x) + 2x^2 \cdot cos\,(ln\,x) \cdot \frac{1}{x}}{5} -$$

$$- \frac{2x \cdot cos\,(ln\,x) - x^2\,sen\,(ln\,x) \cdot \frac{1}{x}}{5} =$$

$$= \frac{4x \cdot sen\,(ln\,x) + 2x \cdot cos\,(ln\,x) - 2x \cdot cos\,(ln\,x) + x \cdot sen\,(ln\,x)}{5} =$$

$$= \frac{5x \cdot sen\,(ln\,x)}{5} = x \cdot sen\,(ln\,x)$$

$$I_2 = \int x \cdot ln\,(x^2+1)\,dx = \frac{x^2}{2}\,ln\,(x^2+1) - \int \frac{x^3}{x^2+1}\,dx =$$

$$= \frac{x^2}{2}\,ln\,(x^2+1) - \int x\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx =$$

$$= \frac{x^2}{2}\,ln\,(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\,ln\,(x^2+1) + C$$

Hallamos la derivada de la función:

$$D\left[\frac{x^2}{2}\,ln\,(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\,ln\,(x^2+1) + C\right] =$$

$$= x\,ln\,(x^2+1) + \frac{x^3}{x^2+1} - x + \frac{x}{x^2+1} = x\,ln\,(x^2+1)$$

$$I_3 = \int x^2 \cdot \ln(2x+1)\, dx = \frac{1}{3} x^3 \ln(2x+1) -$$

$$-\frac{1}{3} \int \frac{2x^3}{2x+1}\, dx = \frac{1}{3} x^3 \ln(2x+1) -$$

$$-\frac{1}{3} \int \left[x^2 - \frac{1}{2} x + \frac{1}{4} - \frac{1}{4} \frac{1}{2x+1}\right] dx =$$

$$= \frac{1}{3} x^3 \ln(2x+1) - \frac{1}{3} \int x^2\, dx + \frac{1}{6} \int x\, dx -$$

$$-\frac{1}{12} \int dx + \frac{1}{24} \int \frac{2}{2x+1}\, dx =$$

$$= \frac{1}{3} x^3 \ln(2x+1) - \frac{1}{9} x^3 + \frac{1}{12} x^2 - \frac{1}{12} x +$$

$$+ \frac{1}{24} \ln(2x+1) + C$$

Hallamos la derivada de la función anterior:

$$D(I_3) = x^2 \ln(2x+1) + \frac{2}{3} \frac{x^3}{2x+1} - \frac{1}{3} x^2 +$$

$$+ \frac{1}{6} x - \frac{1}{12} + \frac{1}{12} \frac{1}{2x+1} = x^2 \ln(2x+1)$$

18. La solución es:

Si $f''(x) = 3x \Rightarrow f'(x) = \frac{3x^2}{2} + C$

Como $f'(0) = 2 \Rightarrow C = 2$, por tanto:

$$f'(x) = \frac{3x^2}{2} + 2 \Rightarrow f(x) = \frac{x^3}{2} + 2x + D$$

Como $f(0) = 1 \Rightarrow D = 1$, luego la función $f(x)$ buscada es:

$$f(x) = \frac{x^3}{2} + 2x + 1$$

19. La función buscada es: $f(x) = \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + 5x$